

# An exact Turán result for tripartite 3-graphs

Adam Sanitt\*      John Talbot†

April 30, 2015

## Abstract

Mantel's theorem says that among all triangle-free graphs of a given order the balanced complete bipartite graph is the unique graph of maximum size. We prove an analogue of this result for 3-graphs. Let  $K_4^- = \{123, 124, 134\}$ ,  $F_6 = \{123, 124, 345, 156\}$  and  $\mathcal{F} = \{K_4^-, F_6\}$ : for  $n \neq 5$  the unique  $\mathcal{F}$ -free 3-graph of order  $n$  and maximum size is the balanced complete tripartite 3-graph  $S_3(n)$  (for  $n = 5$  it is  $C_5^{(3)} = \{123, 234, 345, 145, 125\}$ ). This extends an old result of Bollobás that  $S_3(n)$  is the unique 3-graph of maximum size with no copy of  $K_4^- = \{123, 124, 134\}$  or  $F_5 = \{123, 124, 345\}$ .

## 1 Introduction

If  $r \geq 2$  then an  $r$ -graph  $G$  is a pair  $G = (V(G), E(G))$ , where  $E(G)$  is a collection of  $r$ -sets from  $V(G)$ . The elements of  $V(G)$  are called *vertices* and the  $r$ -sets in  $E(G)$  are called *edges*. The number of vertices is the *order* of  $G$ , while the number of edges, denoted by  $e(G)$ , is the *size* of  $G$ .

Given a family of  $r$ -graphs  $\mathcal{F}$ , an  $r$ -graph  $G$  is  $\mathcal{F}$ -free if it does not contain a subgraph isomorphic to any member of  $\mathcal{F}$ . For an integer  $n \geq r$  we define the *Turán number* of  $\mathcal{F}$  to be

$$\text{ex}(n, \mathcal{F}) = \max\{e(G) : G \text{ an } \mathcal{F}\text{-free } r\text{-graph of order } n\}.$$

The related asymptotic *Turán density* is the following limit (an averaging argument due to Katona, Nemetz and Simonovits [7] shows that it always exists)

$$\pi(\mathcal{F}) = \lim_{n \rightarrow \infty} \frac{\text{ex}(n, \mathcal{F})}{\binom{n}{r}}.$$

---

\*Department of Mathematics, University College London, WC1E 6BT, UK. Email: adam@sanitt.com

†Department of Mathematics, University College London, WC1E 6BT, UK. Email: j.talbot@ucl.ac.uk.

The problem of determining the Turán density is essentially solved for all 2-graphs by the Erdős–Stone–Simonovits Theorem.

**Theorem 1 (Erdős and Stone [5], Erdős and Simonovits [4])** *Let  $\mathcal{F}$  be a family of 2-graphs. If  $t = \min\{\chi(F) : F \in \mathcal{F}\} \geq 2$ , then*

$$\pi(\mathcal{F}) = \frac{t-2}{t-1}.$$

It follows that the set of all Turán densities for 2-graphs is  $\{0, 1/2, 2/3, 3/4, \dots\}$ .

There is no analogous result for  $r \geq 3$  and most progress has been made through determining the Turán densities of individual graphs or families of graphs. A central problem, originally posed by Turán, is to determine  $\text{ex}(n, K_4^{(3)})$ , where  $K_4^{(3)} = \{123, 124, 134, 234\}$  is the complete 3-graph of order 4. This is a natural extension of determining the Turán number of the triangle for 2-graphs, a question answered by Mantel’s theorem [9]. Turán gave a construction that he conjectured to be optimal that has density  $5/9$  but this question remains unanswered despite a great deal of work. The current best upper bound for  $\pi(K_4^{(3)})$  is 0.561666, given by Razborov [11].

A related problem due to Katona is given by considering cancellative hypergraphs. A hypergraph  $H$  is *cancellative* if for any distinct edges  $a, b \in H$ , there is no edge  $c \in H$  such that  $a \triangle b \subseteq c$  (where  $\triangle$  denotes the symmetric difference). For 2-graphs, this is equivalent to forbidding all triangles. For a 3-graph, it is equivalent to forbidding the two non-isomorphic configurations  $K_4^- = \{123, 124, 134\}$  and  $F_5 = \{123, 124, 345\}$ .

An  $r$ -graph  $G$  is *k-partite* if there is a partition of its vertices into  $k$  classes so that all edges of  $G$  contain at most one vertex from each class. It is *complete k-partite* if there is a partition into  $k$  classes such that all edges meeting each class at most once are present. If the partition of the vertices of a complete  $k$ -partite graph is into classes that are as equal as possible in size then we say that  $G$  is *balanced*.

Let  $S_3(n)$  be the complete balanced tripartite 3-graph of order  $n$ .

**Theorem 2 (Bollobás [3])** *For  $n \geq 3$ ,  $S_3(n)$  is the unique cancellative 3-graph of order  $n$  and maximum size.*

This result was refined by Frankl and Füredi [6] and Keevash and Mubayi [8], who proved that  $S_3(n)$  is the unique  $F_5$ -free 3-graph of order  $n$  and maximum size, for  $n$  sufficiently large.

The *blow-up* of an  $r$ -graph  $H$  is the  $r$ -graph  $H(t)$  obtained from  $H$  by replacing each vertex  $a \in V(H)$  with a set of  $t$  vertices  $V_a$  in  $H(t)$  and inserting a complete

$r$ -partite  $r$ -graph between any  $r$  vertex classes corresponding to an edge in  $H$ . The following result is an invaluable tool in determining the Turán density of an  $r$ -graph that is contained in the blow-ups of other  $r$ -graphs:

**Theorem 3 (Brown and Simonovits [1], [2])** *If  $F$  is a  $k$ -graph that is contained in a blow-up of every member of a family of  $k$ -graphs  $\mathcal{G}$ , then  $\pi(F) = \pi(F \cup \mathcal{G})$ .*

Since  $F_5$  is contained in  $K_4^-(2)$ , Theorems 2 and 3 imply that  $\pi(F_5) = 2/9$ .

A natural question to ask is which 3-graphs (that are not subgraphs of blow-ups of  $F_5$ ) also have Turán density  $2/9$ ? Baber and Talbot [2] considered the 3-graph  $F_6 = \{123, 124, 345, 156\}$ , which is not contained in any blow-up of  $F_5$ . Using Razborov's flag algebra framework [10], they gave a computational proof that  $\pi(F_6) = 2/9$ . In this paper, we obtain a new (non-computer) proof of this result. In fact we go further and determine the exact Turán number of  $\mathcal{F} = \{F_6, K_4^-\}$ .

**Theorem 4** *If  $n \geq 3$  then the unique  $\mathcal{F}$ -free 3-graph with  $ex(n, \mathcal{F})$  edges and  $n$  vertices is  $S_3(n)$  unless  $n = 5$  in which case it is  $C_5^{(3)}$ .*

As  $F_6$  is contained in  $K_4^-(2)$ , we have the following corollary to Theorem 3.

**Corollary 5**  $\pi(F_6) = 2/9$ .

## 2 Turán number

*Proof of Theorem 4:* We use induction on  $n$ . Note that the result holds trivially for  $n = 3, 4$ . For  $n = 5$  it is straightforward to check that the only  $\mathcal{F}$ -free 3-graphs with 4 edges are  $S_3(5)$ ,  $\{123, 124, 125, 345\}$  and  $\{123, 234, 345, 451\}$ . Of these the first two are edge maximal while the third can be extended by a single edge to give  $C_5^{(3)}$ . Thus we may suppose that  $n \geq 6$  and the theorem is true for  $n - 3$ .

For  $k \geq 2$  let  $T_k(n)$  be the  $k$ -partite Turán graph of order  $n$ : this is the complete balanced  $k$ -partite graph. We denote the number of edges in  $S_3(n)$  and  $T_k(n)$  by  $s_3(n)$  and  $t_k(n)$  respectively. Let  $G$  be  $\mathcal{F}$ -free with  $n \geq 6$  vertices and  $ex(n, \mathcal{F})$  edges. Since  $S_3(n)$  is  $\mathcal{F}$ -free we have  $e(G) \geq s_3(n)$ .

The inductive step proceeds as follows: select a special edge  $abc \in E(G)$  (precisely how we choose this edge will be explained in Lemma 6 below). For  $0 \leq i \leq 3$  let  $f_i$  be the number of edges in  $G$  meeting  $abc$  in exactly  $i$  vertices.

By our inductive hypothesis we have

$$e(G) = f_0 + f_1 + f_2 + f_3 \leq \text{ex}(n-3, \mathcal{F}) + f_1 + f_2 + 1. \quad (1)$$

Note that unless  $n-3 = 5$  our inductive hypothesis says that  $\text{ex}(n-3, \mathcal{F}) = s_3(n-3)$  with equality iff  $G - \{a, b, c\} = S_3(n-3)$ . For the moment we will assume that  $n \neq 8$  and so we have the following bound

$$e(G) \leq s_3(n-3) + f_1 + f_2 + 1, \quad (2)$$

with equality iff  $G - \{a, b, c\} = S_3(n-3)$ .

Let  $V^- = V(G) - \{a, b, c\}$ . For each pair  $xy \in \{ab, ac, bc\}$  define  $\Gamma_{xy} = \{z \in V^- : xyz \in E(G)\}$  and let  $\Gamma_{abc} = \Gamma_{ab} \cup \Gamma_{ac} \cup \Gamma_{bc}$  be the *link-neighbourhood* of  $abc$ . Note that since  $G$  is  $K_4^-$ -free this is a disjoint union, so

$$f_2 = |\Gamma_{ab}| + |\Gamma_{ac}| + |\Gamma_{bc}| = |\Gamma_{abc}|.$$

For  $x \in \{a, b, c\}$  define  $L(x)$  to be the *link-graph* of  $x$ , so  $V(L(x)) = V^-$  and  $E(L(x)) = \{yz \subset V^- : xyz \in E(G)\}$ . The *link-graph of the edge  $abc$*  is the edge labelled graph  $L_{abc}$  with vertex set  $V^-$  and edge set  $L(a) \cup L(b) \cup L(c)$ . The label of an edge  $yz \in E(L_{abc})$  is  $l(yz) = \{x \in \{a, b, c\} : xyz \in E(G)\}$ . The *weight* of an edge  $yz \in L_{abc}$  is  $|l(yz)|$  and the weight of  $L_{abc}$  is  $w(L_{abc}) = \sum_{yz \in L_{abc}} |l(yz)|$ . Note that  $f_1 = w(L_{abc})$ .

By a subgraph of  $L_{abc}$  we mean an ordinary subgraph of the underlying graph where the labels of edges are non-empty subsets of the labels of the edges in  $L_{abc}$ . For example if  $xy \in E(L_{abc})$  has  $l(xy) = ab$  then in any subgraph of  $L_{abc}$  containing the edge  $xy$  it must have label  $a, b$  or  $ab$ .

A triangle in  $L_{abc}$  is said to be *rainbow* iff all its edges have weight one and are labelled  $a, b, c$ . Given an edge labelled subgraph  $H$  of  $L_{abc}$  and an (unlabelled) graph  $G$  we say that  $H$  is a *rainbow  $G$*  if all of the edges in  $H$  have weight 1 and all the triangles in  $H$  are rainbow.

The following lemma provides our choice of edge  $abc$ .

**Lemma 6** *If  $G$  is an  $\mathcal{F}$ -free 3-graph with  $n \geq 6$  vertices and  $\text{ex}(n, \mathcal{F})$  edges then there is an edge  $abc \in E(G)$  such that*

$$w(L_{abc}) + |\Gamma_{abc}| \leq t_3(n-3) + n - 3,$$

*with equality iff  $L_{abc}$  is a rainbow  $T_3(n-3)$  and  $\Gamma_{abc} = V^-$ .*

Underlying all our analysis are some simple facts regarding  $\mathcal{F}$ -free 3-graphs that are contained in Lemmas 7 and 8.

**Lemma 7** *If  $G$  is  $\mathcal{F}$ -free and  $abc \in E(G)$  then the following configurations cannot appear as subgraphs of  $L_{abc}$ . Moreover any configuration that can be obtained from one described below by applying a permutation to the labels  $\{a, b, c\}$  must also be absent.*

- (F<sub>6</sub>-1) *The triangle  $xy, xz, yz$  with  $l(xy) = l(xz) = a$  and  $l(yz) = b$ .*
- (F<sub>6</sub>-2) *The pair of edges  $xy, xz$  with  $l(xy) = ab$  and  $l(xz) = c$ .*
- (F<sub>6</sub>-3) *A vertex  $x \in \Gamma_{ab}$  and edges  $xy, yz$  with labels  $l(xy) = c$  and  $l(yz) = a$ .*
- (F<sub>6</sub>-4) *A vertex  $x \in \Gamma_{ab}$  and edges  $xy, yz, zw$  with labels  $l(xy) = l(zw) = a$  and  $l(yz) = b$ .*
- (F<sub>6</sub>-5) *Vertices  $x \in \Gamma_{ac}, y \in \Gamma_{bc}, z \in \Gamma_{ab}$  and the edge  $xy$  with label  $l(xy) = b$ .*
- (K<sub>4</sub><sup>-</sup>-1) *The triangle  $xy, xz, yz$  with  $l(xy) = l(xz) = l(yz) = a$ .*
- (K<sub>4</sub><sup>-</sup>-2) *The vertex  $x \in \Gamma_{ab}$  and edge  $xy$  with label  $l(xy) = ab$ .*
- (K<sub>4</sub><sup>-</sup>-3) *The vertices  $x, y \in \Gamma_{ab}$  and edge  $xy$  with label  $l(xy) = a$ .*

**Lemma 8** *If  $G$  is  $\mathcal{F}$ -free and  $abc \in E(G)$  then the link-graph and link-neighbourhood satisfy:*

- (i) *The only triangles in  $L_{abc}$  are rainbow.*
- (ii) *The only  $K_4$ s in  $L_{abc}$  are rainbow.*
- (iii)  *$L_{abc}$  is  $K_5$ -free.*
- (iv) *If  $xy \in E(L_{abc})$  has  $l(xy) = abc$  then  $x$  and  $y$  meet no other edges in  $L_{abc}$  and  $x, y \notin \Gamma_{abc}$ .*
- (v) *If  $V_{abc}^4 = \{x \in V^- : \text{there is a } K_4 \text{ containing } x\}$  then  $\Gamma_{abc} \cap V_{abc}^4 = \emptyset$ .*
- (vi) *There are no edges in  $L_{abc}$  between  $\Gamma_{abc}$  and  $V_{abc}^4$ .*
- (vii) *If  $x \in V_{abc}^4$  then  $|l(xy)| \leq 1$  for all  $y \in V^-$ .*
- (viii) *If  $x \in \Gamma_{ac}, y \in \Gamma_{bc}$  and  $l(xy) = ab$ , then  $\Gamma_{bc} = \emptyset$ . Moreover, if  $xz \in E(L_{abc})$  with  $z \neq y$  then  $z \notin \Gamma_{abc}$  and  $l(xz) = a$ , while if  $yz \in E(L_{abc})$  with  $z \neq x$  then  $z \notin \Gamma_{abc}$  and  $l(yz) = b$ .*
- (ix) *If  $xy, xz \in E(L_{abc})$ ,  $l(xy) = ab$  and  $z \in \Gamma_{abc}$  then  $|l(xz)| \leq 1$ .*

We also require the following identities, that are easy to verify.

**Lemma 9** *If  $n \geq k \geq 3$  then*

$$(i) \quad s_3(n) = s_3(n-3) + t_3(n-3) + n - 2.$$

$$(ii) \quad t_3(n) = t_3(n-3) + 2n - 3.$$

$$(iii) \quad t_3(n) = t_3(n-2) + n - 1 + \lfloor n/3 \rfloor.$$

$$(iv) \quad t_k(n) = t_k(n-1) + n - \lceil n/k \rceil.$$

Let  $abc \in E(G)$  be a fixed edge given by Lemma 6.

By assumption  $e(G) \geq s_3(n)$  so Lemma 9 (i) and Lemma 6 together with the bound on  $e(G)$  given by (2) imply that  $e(G) = s_3(n)$  and hence  $G - \{a, b, c\} = S_3(n-3)$ ,  $L_{abc}$  is a rainbow  $T_3(n-3)$  and  $\Gamma_{abc} = V^-$ . To complete the proof we need to show that  $G = S_3(n)$ . First note that since  $L_{abc}$  is a rainbow  $T_3(n-3)$  and  $\Gamma_{abc} = V^-$ , Lemma 8 (i) and Lemma 7( $F_6$ -3) imply that no vertex in  $\Gamma_{ab}$  is in an edge with label  $c$  and similarly for  $\Gamma_{ac}, \Gamma_{bc}$ . Hence  $L_{abc}$  is the complete tripartite graph with vertex classes  $\Gamma_{ab}, \Gamma_{ac}$  and  $\Gamma_{bc}$  and the edges between any two parts are labelled with the common label of the parts (e.g. all edges from  $\Gamma_{ab}$  to  $\Gamma_{ac}$  receive label  $a$ ). So  $L_{abc}$  is precisely the link graph of an edge  $abc \in S_3(n)$ .

In order to deduce that  $G = S_3(n)$  we need to show that  $G - \{a, b, c\} = S_3(n-3)$  has the same tripartition as  $L_{abc}$ . This is straightforward: any edge  $xyz \in E(G - \{a, b, c\})$  not respecting the tripartition of  $L_{abc}$  meets one of the parts at least twice. But if  $x, y, z \in \Gamma_{ab}$  then  $|\Gamma_{ac}| \geq 2$  so let  $u \in \Gamma_{ac}$ . Setting  $a = 1, b = 2, x = 3, y = 4, z = 5, u = 6$  gives a copy of  $F_6$ . If  $x, y \in \Gamma_{ab}$  and  $z \in \Gamma_{ac}$  then  $a = 1, x = 3, y = 4, z = 2$  gives a copy of  $K_4^-$ .

Hence  $G = S_3(n)$  and the proof is complete in the case  $n \neq 8$ .

For  $n = 8$  we note that if  $G - \{a, b, c\}$  is  $F_5$ -free then Theorem 2 implies that the result follows as above, so we may assume that  $G - \{a, b, c\}$  contains a copy of  $F_5$ . In this case it is sufficient to show that  $e(G) \leq 17 < 18 = s_3(8)$ .

If  $V(G - \{a, b, c\}) = \{s, t, u, v, w\}$  then we may suppose that  $stu, stv, uvw, abc \in G$ . Since  $G$  is  $K_4^-$ -free it does not contain  $suv$  or  $tuv$ . Moreover it contains at most 3 edges from  $\{u, v, w\}^{(2)} \times \{a, b, c\}$  and at most 5 edges from  $\{s, t, u, v, w\} \times \{a, b, c\}^{(2)}$ . Since  $G$  is  $F_6$ -free it contains no edges from  $\{s, t\} \times \{w\} \times \{a, b, c\}$ .

The only potential edges we have yet to consider are those in  $\{st, su, tu, sv, tv\} \times \{w, a, b, c\}$ . Since  $G$  is  $K_4^-$ -free it contains at most 2 edges from  $std, sud, tud, svd, tvd$ , for any  $d \in \{w, a, b, c\}$ . Moreover, since  $G$  is  $F_6$ -free, if it contains 2 such edges for a fixed  $d$  then it can contain at most 3 such edges in total for the other choices of  $d$ . Hence at most 5 such edges are present.

Thus in total  $e(G) \leq 4 + 3 + 5 + 5 = 17$ , as required.  $\square$

In order to prove Lemma 6 we first need an edge with large link-neighbourhood.

**Lemma 10** *If  $G$  is  $K_4^-$ -free 3-graph of order  $n$  with  $s_3(n)$  edges, then there is an edge  $abc \in E(G)$  with  $|\Gamma_{abc}| \geq n - \lfloor n/3 \rfloor - 3$ .*

*Proof of Lemma 10:* Let  $G$  be  $K_4^-$ -free with  $n$  vertices and  $s_3(n)$  edges. For  $x, y \in V(G)$  let  $d_{xy} = |\{x : xyz \in E(G)\}|$ . If  $uvw \in E(G)$  then  $\Gamma_{uvw} = \Gamma_{uv} \cup \Gamma_{uw} \cup \Gamma_{vw}$  is a union of pairwise disjoint sets and  $|\Gamma_{uvw}| = d_{uv} + d_{uw} + d_{vw} - 3$ . Thus if the lemma fails to hold then for every edge  $uvw \in E(G)$  we have  $d_{uv} + d_{uw} + d_{vw} \leq n - \lfloor n/3 \rfloor - 1$ . Note that since  $\sum_{xy \in \binom{V}{2}} d_{xy} = 3e(G)$ , convexity implies that

$$e(G)(n - \lfloor \frac{n}{3} \rfloor - 1) \geq \sum_{uvw \in E(G)} d_{uv} + d_{uw} + d_{vw} = \sum_{xy \in \binom{V}{2}} d_{xy}^2 \geq \frac{9e^2(G)}{\binom{n}{2}}.$$

Thus

$$e(G) \leq \frac{1}{18}n(n-1)(n - \lfloor n/3 \rfloor - 1).$$

But it is easy to check that this is less than  $s_3(n)$ .  $\square$

Our next objective is to describe various properties of the link-graph  $L_{abc}$  and link-neighbourhood  $\Gamma_{abc}$ .

Lemma 8 (v) allows us to partition the vertices of  $L_{abc}$  as  $V^- = \Gamma_{abc} \cup V_{abc}^4 \cup R_{abc}$ , where  $V_{abc}^4 = \{x \in V^- : \text{there is a } K_4 \text{ containing } x\}$  and  $R_{abc} = V^- - (\Gamma_{abc} \cup V_{abc}^4)$ . To prove Lemma 6 we require the following result to deal with the part of  $L_{abc}$  not meeting any copies of  $K_4$ .

**Lemma 11** *Let  $H$  be a subgraph of  $L_{abc}$  with  $s \geq 3$  vertices satisfying  $V(H) \cap V_{abc}^4 = \emptyset$ . If  $H_\Gamma = V(H) \cap \Gamma_{abc}$  and  $|H_\Gamma| \geq s - \lfloor s/3 \rfloor - 1$  then*

$$w(H) + |H_\Gamma| \leq t_3(s) + s,$$

*with equality iff  $H_\Gamma = V(H)$  and  $H$  is a rainbow  $T_3(s)$ .*

*Proof of Lemma 6:* Let  $G$  be  $\mathcal{F}$ -free with  $n \geq 6$  vertices and  $\text{ex}(n, \mathcal{F})$  edges. By Lemma 10 we can choose an edge  $abc \in E(G)$  such that  $|\Gamma_{abc}| \geq n - \lfloor n/3 \rfloor - 3$ . Let  $V^- = \Gamma_{abc} \cup R_{abc} \cup V_{abc}^4$  be the partition of  $V^-$  given by Lemma 8 (v). If  $s = |V^-|$ ,  $j = |\Gamma_{abc}|$ ,  $k = |R_{abc}|$  and  $l = |V_{abc}^4|$  then  $n - 3 = s = j + k + l$  and  $j \geq s - \lfloor s/3 \rfloor - 1 \geq j + k - \lfloor (j + k)/3 \rfloor - 1$ . We can apply Lemma 11 to  $H = L_{abc}[\Gamma_{abc} \cup R_{abc}]$ , to deduce that

$$w(L_{abc}[\Gamma_{abc} \cup R_{abc}]) + |\Gamma_{abc}| \leq t_3(j + k) + j + k,$$

with equality iff  $R_{abc} = \emptyset$  and  $L_{abc}[\Gamma_{abc}]$  is a rainbow  $T_3(j + k)$ . Now if  $L_{abc}$  is  $K_4$ -free then  $V_{abc}^4 = \emptyset$  and the proof is complete, so suppose there is a  $K_4$  in  $L_{abc}$ . In this case  $4 \leq |V_{abc}^4| \leq n - 3 - |\Gamma_{abc}| \leq \lfloor n/3 \rfloor$ , so  $n \geq 12$ .

We now need to consider the edges in  $L_{abc}$  meeting  $V_{abc}^4$ . By Lemma 8 (iii) we know that  $L_{abc}$  is  $K_5$ -free, while Lemma 8 (vii) says that  $V_{abc}^4$  meets no edges of weight 2 or 3, so by Turán's theorem  $w(L_{abc}[V_{abc}^4]) \leq t_4(l)$ .

Lemma 8 (vi) implies that there are no edges from  $\Gamma_{abc}$  to  $V_{abc}^4$  so the total weight of edges between  $\Gamma_{abc} \cup R_{abc}$  and  $V_{abc}^4$  is at most  $kl$ . Thus

$$w(L_{abc}) + |\Gamma_{abc}| \leq t_3(j+k) + j+k + t_4(l) + kl.$$

Finally Lemma 12 with  $s = n - 3$  implies that

$$w(L_{abc}) + |\Gamma_{abc}| \leq t_3(n-3) + n-3,$$

with equality iff  $R_{abc} = V_{abc}^4 = \emptyset$  and  $L_{abc}$  is a rainbow  $T_3(n-3)$  as required.  $\square$

**Lemma 12** *If  $j, k, l \geq 0$  are integers satisfying  $j+k+l = s \geq 5$  and  $j \geq s - \lfloor s/3 \rfloor - 1$  then*

$$t_3(j+k) + t_4(l) + j+k + kl \leq t_3(s) + s, \quad (3)$$

*with equality iff  $l = 0$ .*

*Proof of Lemma 12:* If  $l = 0$  then the result clearly holds, so suppose that  $l \geq 1$ ,  $j+k+l = s \geq 5$  and  $j \geq s - \lfloor s/3 \rfloor - 1$ . Let  $f(j, k, l)$  be the LHS of (3). We need to check that  $\Delta(j, k, l) = f(j, k+1, l-1) - f(j, k, l) > 0$ . Using Lemma 9 (iv) we have

$$\begin{aligned} \Delta(j, k, l) &= j - \lceil (j+k+1)/3 \rceil + \lceil l/4 \rceil + 1 \\ &= j + \lceil l/4 \rceil - \lfloor (j+k)/3 \rfloor. \end{aligned}$$

So it is sufficient to check that  $j + l/4 > (j+k)/3$ . This follows easily from  $j \geq s - \lfloor s/3 \rfloor - 1$ ,  $k \leq \lfloor s/3 \rfloor + 1$ ,  $l \geq 1$  and  $s \geq 5$ .  $\square$

*Proof of Lemma 11:* We prove this by induction on  $s \geq 3$ . The result holds for  $s = 3, 4$  (see the end of this proof for the tedious details) so suppose that  $s \geq 5$  and the result holds for  $s-2$ .

Let  $H$  be a subgraph of  $L_{abc}$  with  $s \geq 5$  vertices satisfying  $V(H) \cap V_{abc}^4 = \emptyset$ . Let  $H_\Gamma = V(H) \cap \Gamma_{abc}$  and suppose that  $|H_\Gamma| \geq s - \lfloor s/3 \rfloor - 1$ .

Note that if  $H$  contains no edges of weight 2 or 3 then the result follows directly from Turán's theorem and Lemma 8 (i), so we may suppose there are edges of weight 2 or 3. With this assumption it is sufficient to show that

$$w(H) + |H_\Gamma| \leq t_3(s) + s - 1.$$

By Lemma 9 (iii) this is equivalent to showing that the following inequality holds:

$$w(H) + |H_\Gamma| \leq t_3(s-2) + 2s-2 + \lfloor s/3 \rfloor \quad (4)$$



**Case (i):** There exists an edge of weight 3,  $l(xy) = abc$ .

Lemma 8 (iv) implies that  $x, y \notin H_\Gamma$  and  $x, y$  meet no other edges in  $H$ , so we can apply the inductive hypothesis to  $H' = H - \{x, y\}$  to obtain

$$w(H) + |H_\Gamma| \leq w(H') + |H'_\Gamma| + 3 \leq t_3(s-2) + s - 2 + 3.$$

Hence (4) holds as required. So we may suppose that  $H$  contains no edges of weight 3.

**Case (ii):** The only edges of weight 2 are contained in  $H_\Gamma$

Let  $xy \in E(H)$  have weight 2, say  $l(xy) = ab$ . Now Lemma 7 ( $K_4^-$ -2) implies that  $x, y \notin \Gamma_{ab}$ , while Lemma 7 ( $K_4^-$ -3) implies that  $x, y$  cannot both belong to  $\Gamma_{ac}$  or  $\Gamma_{bc}$  so we may suppose that  $x \in \Gamma_{ac}$  and  $y \in \Gamma_{bc}$ . Lemma 8 (viii) implies that  $x, y$  have no more neighbours in  $H_\Gamma$ . If  $H_\Gamma = V(H)$  then we can apply the inductive hypothesis to  $H' = H - \{x, y\}$  to obtain

$$w(H) + |H_\Gamma| \leq t_3(s-2) + s - 2 + 2 + 2,$$

in which case (4) holds, so suppose  $V(H) \neq H_\Gamma$ .

Let  $z \in V(H) - H_\Gamma$  be a neighbour of  $x$  in  $H$  if one exists otherwise let  $z$  be any vertex in  $V(H) - H_\Gamma$ . By our assumption that all edges of weight 2 are contained in  $H_\Gamma$ ,  $z$  meets no edges of weight 2. Moreover, by Lemma 8 (viii), all edges containing  $x$  (except  $xy$ ) have label  $b$ , so  $x$  is not in any triangles in  $H$ . Hence  $x$  and  $z$  have no common neighbours in  $H$  and so the total weight of edges meeting  $\{x, z\}$  is at most  $2 + 1 + s - 3$  (if  $xz$  is an edge) and at most  $2 + s - 2$  otherwise. Applying our inductive hypothesis to  $H' = H - \{x, z\}$  we have

$$w(H) + |H_\Gamma| \leq t_3(s-2) + s - 2 + 1 + s,$$

and (4) holds.

**Case (iii):** There is an edge of weight 2 meeting  $V(H) - H_\Gamma$ .

So suppose that  $xy \in E(H)$ ,  $l(xy) = ab$  and  $y \notin H_\Gamma$ . Lemma 8 (ix) implies that for any  $z \in H_\Gamma$  we have  $|l(xz)|, |l(yz)| \leq 1$ . Let  $\gamma_{xy} = |\{x, y\} \cap H_\Gamma| \leq 1$ . Thus, since  $xy$  is not in any triangles, the total weight of edges meeting  $\{x, y\}$  is at most

$$2 + s - 2 + |V(H) - H_\Gamma| - (2 - \gamma_{xy}).$$

Applying the inductive hypothesis to  $H' = H - \{x, y\}$  we have

$$w(H) + |H_\Gamma| \leq t_3(s-2) + s - 2 + s + s - |H_\Gamma| - 2 + 2\gamma_{xy},$$

with equality holding only if  $|H'_\Gamma| = s - 2$ . Now  $|H_\Gamma| \geq s - \lfloor s/3 \rfloor - 1$  implies that

$$w(H) + |H_\Gamma| \leq t_3(s-2) + 2s - 3 + \lfloor s/3 \rfloor + 2\gamma_{xy}, \quad (5)$$

with equality only if  $|H'_\Gamma| = s - 2$  and  $|H_\Gamma| = s - \lfloor s/3 \rfloor - 1$ . If  $\gamma_{xy} = 0$  then (4) holds as required, so suppose  $\gamma_{xy} = 1$ . In this case (4) holds, unless (5) holds with equality. But if (5) is an equality then  $|H_\Gamma| = |H'_\Gamma| + 1 = s - 1$ , while  $|H_\Gamma| = s - \lfloor s/3 \rfloor - 1$ , which is impossible for  $s \geq 3$ .

We finally need to verify the cases  $s = 3, 4$ . It is again sufficient to prove that if  $H$  contains edges of weight 2 or 3 then  $w(H) + |H_\Gamma| \leq t_3(s) + s - 1$ , thus we need to show that  $w(H) + |H_\Gamma|$  is at most 5 if  $s = 3$  and at most 8 if  $s = 4$ .

We note that argument in Case (i) above implies that if  $H$  contains an edge of weight 3 then  $|H_\Gamma| \leq s - 2$  and  $w(H) \leq 3 + 3\binom{s-2}{2}$ , so if  $s = 3$  then  $w(H) + |H_\Gamma| \leq 4$  and if  $s = 4$  then  $w(H) + |H_\Gamma| \leq 8$  so the result holds. So we may suppose there are no edges of weight 3.

Now let  $xy$  be an edge of weight 2. Using the fact that  $xy$  is not in any triangles and Lemma 8 (viii) and (ix) we find that for  $s = 3$  we have  $w(H) + |H_\Gamma| \leq 2 + 3 - |H_\Gamma|$ , while for  $s = 4$  we have  $w(H) + |H_\Gamma| \leq 2 + 6 - |H_\Gamma|$ , so the result holds.  $\square$

Finally we need to establish our two structural lemmas.

*Proof of Lemma 7:* In each case we describe a labelling of the vertices of the given configuration to show that if it is present then  $G$  is not  $\mathcal{F}$ -free.

( $F_6$ -1)  $a = 1, b = 5, c = 6, x = 2, y = 3, z = 4$ .

( $F_6$ -2)  $a = 3, b = 4, c = 5, x = 1, y = 2, z = 6$ .

( $F_6$ -3)  $a = 1, b = 2, c = 3, x = 4, y = 5, z = 6$ .

( $F_6$ -4)  $a = 1, b = 3, x = 2, y = 4, z = 5, w = 6$ .

( $F_6$ -5)  $a = 5, b = 1, c = 3, x = 4, y = 2, z = 6$ .

( $K_4^-$ -1)  $a = 1, x = 2, y = 3, z = 4$ .

( $K_4^-$ -2)  $a = 3, b = 4, x = 1, y = 2$ .

( $K_4^-$ -3)  $a = 1, b = 2, x = 3, y = 4$ .  $\square$

*Proof of Lemma 8:* We will make repeated use of Lemma 7.

(i) This follows immediately from ( $F_6$ -1) and ( $K_4^-$ -1).

(ii) This follows immediately from (i): if  $uvw x$  is a copy of  $K_4$  then we may suppose  $l(uv) = a, l(uw) = b, l(vw) = c$ , thus  $l(ux) = c$  (otherwise (i) would be violated) continuing we see that  $uvw x$  must be rainbow.

(iii) This follows immediately from (ii): if  $xyzuv$  is a copy of  $K_5$  then by (ii) we may suppose that  $l(xy), l(xz), l(xu), l(xv)$  are all distinct single labels from  $\{a, b, c\}$  but this is impossible since there are only 3 labels in total.

(iv) This follows immediately from  $(F_6-2)$  and  $(K_4^- -2)$ .

(v) If  $x$  is in a  $K_4$  then by (ii) it lies in edges with labels  $a, b, c$ , so  $(F_6-3)$  implies that  $x \notin \Gamma_{abc}$ .

(vi) If  $x \in \Gamma_{abc}$ , say  $x \in \Gamma_{ab}$ , and  $y \in V_{abc}^4$  with  $xy \in E(L_{abc})$  then  $(F_6-3)$  implies that  $l(xy) \neq c$ , while  $(F_6-4)$  implies that  $l(xy) \neq a, b$  (since there are  $t, u, v, w$  such that  $l(yt) = b, l(tu) = a$  and  $l(yv) = a, l(vw) = b$ ).

(vii) This follows immediately from the fact that all  $v \in V_{abc}^4$  meet edges with labels  $a, b, c$  and  $(F_6-2)$ .

(viii)  $(F_6-5)$  implies that  $\Gamma_{bc} = \emptyset$ . If  $xz \in E(L_{abc})$  then  $(F_6-3)$  implies that  $l(xz) = a$ . Now  $(K_4-3)$  implies that  $z \notin \Gamma_{ac}$  while  $(F_6-3)$  implies that  $z \notin \Gamma_{bc}$ . Hence  $z \notin \Gamma_{abc}$ . Similarly if  $yz \in E(L_{abc})$  then  $l(yz) = b$  and  $z \notin \Gamma_{abc}$ .

(ix) If  $x \in \Gamma_{abc}$  or  $y \in \Gamma_{abc}$  then this follows directly from (viii) so suppose that  $x, y \notin \Gamma_{abc}$ ,  $l(xy) = ab$  and  $|l(xz)| = 2$ . In this case,  $(F_6-2)$  implies that  $l(xz) = ab$  so  $(K_4-2)$  implies that  $z \in \Gamma_{ac} \cup \Gamma_{bc}$ . But then  $(F_6-3)$  is violated. Hence  $|l(xz)| \leq 1$ .  $\square$

### 3 Conclusion

Many Turán-type results have associated “stability” versions, and we were able to obtain such a result. For reasons of length we state it without proof.

**Theorem 13** *For any  $\epsilon > 0$  there exist  $\delta > 0$  and  $n_0$  such that the following holds: if  $H$  is an  $\mathcal{F}$ -free 3-graph of order  $n \geq n_0$  with at least  $(1 - \delta)s_3(n)$  edges, then there is a partition of the vertex set of  $H$  as  $V(H) = U_1 \cup U_2 \cup U_3$  so that all but at most  $\epsilon n^3$  edges of  $H$  have one vertex in each  $U_i$ .*

### References

- [1] W. G. Brown and M. Simonovits, *Digraph extremal problems, hypergraph extremal functions, and the densities of graph structures*, Disc. Math. **48** 147–162 (1984).
- [2] R. Baber and J. Talbot, *New Turán densities for 3-graphs*, Electron. J. Combin. **19** (2) P22 (2012).
- [3] B. Bollobás, *Three-graphs without two triples whose symmetric difference is contained in a third*, Disc. Math. **8** 21–24, (1974).
- [4] P. Erdős and M. Simonovits, *A limit theorem in graph theory*, Studia Sci. Math. Hung. Acad. **1** 51–57, (1966).

- [5] P. Erdős and A.H. Stone, On the structure of linear graphs, Bull. Amer. Math. Soc. 52 (1946) 1087–1091.
- [6] P. Frankl and Z. Füredi, *A new generalization of the Erdős–Ko–Rado theorem*, Combinatorica **3** 341–349, (1983).
- [7] G. Katona, T. Nemetz and M. Simonovits, *On a problem of Turán in the theory of graphs*, Mat. Lapok **15**, 228–238, (1964).
- [8] P. Keevash and D. Mubayi, *Stability theorems for cancellative hypergraphs*, J. Combin. Theory Ser. B **92** 163–175 (2004).
- [9] V. W. Mantel, *Problem 28*, Wiskundige Opgaven 10, (1907), 60–61.
- [10] A. A. Razborov, *Flag Algebras*, Journal of Symbolic Logic, **72** (4) 1239–1282, (2007).
- [11] A. A. Razborov, *On 3-hypergraphs with forbidden 4-vertex configurations*, in SIAM J. Disc. Math. **24**, (3) 946–963 (2010).
- [12] P. Turán, On an extremal problem in graph theory, Mat. Fiz. Lapok 48 (1941) 436–452 [in Hungarian].